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ON THE NUMERICAL SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEM FOR THE HELMHOLTZ TYPE EQUATION BY FINITE DIFFERENCE METHOD WITH NON-REGULAR STEP LENGTH BETWEEN NODES

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***Abstract*— In the present research work, we presented an algorithm using finite difference wit h non-uniform step length i.e. variable step length, for the approximate numerical solution of the second-order differential equation and corresponding boundary value problems in ODEs. We have investigated under appropriate conditions the convergence of the proposed algorithm. We established that the proposed algorithm converges and the order of the convergence/accuracy of the algorithm is at least cubic. We tested the proposed algorithm for the numerical solution of several model problems. The numerical results obtained for these model problems with known/constructed exact solution, justify the theoretical conclusions of the proposed algorithm. The computational results obtained for these model problems suggest that the method is efficient and accurate.**

Keywords*— Boundary Value Problem, Convergence of the Method, Cubic Order, Finite Difference Method, Non-uniform Step Length.

I. INTRODUCTION

For the development of the algorithm, we considered the boundary value problem of the following form

$$u''(x) + K^2u(x) = f(x,u),$$
$$a < x < b \qquad (1)$$

subject to the boundary conditions

$$u(a) = \alpha \text{ and } u(b) = \beta,$$

where K, α and β are constants. Under the following assumptions and refer literature in [1,2,3], we assume the existence and uniqueness of the solution $u(x)$ for the problem (1):

(i) $f(x, u)$ is continuous in $[a, b] \times \mathbb{R}$

(ii) $\frac{\partial f}{\partial u}$ exist and is continuous

(iii) $\frac{\partial f}{\partial u} \geq 0$

Ordinary differential equations find their application in many fields of engineering and science. A specific application of ODE is the modelling of wave problems. For in- stance, the Helmholtz equation in one dimension is used to study acoustic phenomena in aerodynamics [4], underwater acoustic [5,6] and electromagnetic applications [7]. Solving these boundary value problems with a higher value of K in scientific computation is a challenging task. As of late, there has been a lot of interest shown by the computational mathematician in developing methods/ algorithms for the approximate numerical solution of Helmholtz type equation as two-point boundaryvalue problems with a higher value of K.

For the numerical solution of the boundary value problem for the Helmholtz equation, in general, a finite difference method [8] is used. However, there are some other methods reported in the literature such as finite element [10], finite-infinite element [11] and some iterative methods [12] and references therein.

In the present work, we shall develop an algorithm using finite differences for the approximate numerical solution of the problems (1). We shall discuss the order and accuracy of the developed algorithm under appropriate conditions which is otherwise at least three. To demonstrate the efficiency and accuracy of the proposed method, we shall perform numerical experiments with model problems and discussnumerical results.

In this article, we have presented our work as follows. An algorithm using finite differences will be presented in the next section, in Section III we will outline the derivation of the algorithm. In Section IV, calculate the truncation error in the proposed method and Section V, we have discussed convergence and the order of the accuracy. The application of the proposed method, i.e. numerical experiments on the several model problems and illustrative numerical results have been produced to show the convergence in Section VI. In the last section VII, there are a discussion and conclusion on the performance of the proposed method.

II. THE FINITE DIFFERENCE METHOD

We define N finite numbers of nodal points in $[a,b]$, the domain in which the solution of the problem (1) is desired. Let $a \leq x_0 < x_1 < x_2 < .... < x_{N+1} = b$

i.e., $x_{i+1} = x_i + h_{i+1}$, $i = 0(1)N$ and h_i, the variable step length. We wish to determine the approximation of the analytical solution $u(x)$ of the problem (1) at the nodal points x_i, $i = 1, 2, .., N$. We denote the approximation of $u(x)$ at the node $x = x_i$ as u_i for each $i = 1, 2,.., N$. Also, we denote the approximate value of the forcing function $f(x, u(x))$ at node $x = x_i$ as f_i for each $i = 1, 2, N$. We shall follow same notation in defining other notations this article i.e., $f_{i\pm1}$ *and* $u_{i\pm1}$. Thus, using these notations we rewrite the differential equation (1) at themesh point $x = x_i$,

$$u_i'' + K^2 u_i = f(x_i, u_i) \qquad (2)$$

Following the ideas in [8,9], we propose a three points cubic order finite difference method for the numerical solution of the problem (1),

$$a_{2i}u_{i+1} + a_{1i}u_i + a_{0i}u_{i-1} = c_{2i}f_{i+1} + a_{1i}f_i + a_{0i}f_{i-1}, \quad i = 1,2,\dots N. \qquad (3)$$

were

$$a_{2i} = 24(6 - K^2h_i^2),$$

$$c_{2i} = 2h_i^2\Big(r_i(6 - r_iK^2h_i^2) + 6(r_i^2 - 1)\Big),$$

$$a_{1i} = (1 + r_i)(24K^2h_i^2(1 + r_i)^2 - 6r_iK^4h_i^4(1 + r_i + r_i^2) + r_i^3K^6h_i^6 - 144),$$

$$c_{1i} = (1 + r_i)h_i^2\Big(12(r_i^2 + 3r_i + 1) - r_iK^2h_i^2(6r_i^2 + r_i(4 - K^2h_i^2) + 6)\Big),$$

$$a_{0i} = 24r_i(6 - K^2h_i^2r_i^2),$$

$$c_{0i} = 2r_ih_i^2\Big(r_i(6 - r_iK^2h_i^2) + 6(r_i^2 - 1)\Big).$$

III. DERIVATION OF THE METHOD

In this section, we outline the derivation of the algorithm. To develop an algorithm, we discretize problem (1) at the nodal point x_i. We approximate the differential equation (2) by following the difference formula,

$$a_{2i}u_{i+1} + a_{1i}u_i + a_{0i}u_{i-1} = c_{2i}f_{i+1} + c_{1i}f_i + c_{0i}f_{i-1}, \quad i = 1,2,\dots N. \qquad (4)$$

where the coefficients a_{0i}, a_{1i}, a_{2i}, c_{0i}, c_{1i}, c_{2i} are function of r_i and $r_i = \frac{h_{i+1}}{h_i}$. We can determine these coefficients by the method of Taylor series expansion and comparing the various coefficients in the expansion. So let us write terms $u_{i\pm1}$ and $f_{i\pm1}$ in Taylor series about x_i and comparing the coefficients of h^p, $p = 0, 1, .., 4$. We have obtained a system of linear equations in coefficients a_{0i}, a_{1i}, a_{2i}, c_{0i}, c_{1i}, c_{2i}. After solving the system of equations, the coefficients are

$$a_{2i} = 24(6 - K^2h_i^2),$$

$$c_{2i} = 2h_i^2\Big(r_i(6 - r_iK^2h_i^2) + 6(r_i^2 - 1)\Big),$$

$$a_{1i} = (1 + r_i)(24K^2h_i^2(1 + r_i)^2 - 6r_iK^4h_i^4(1 + r_i + r_i^2) + r_i^3K^6h_i^6 - 144),$$

$$c_{1i} = (1 + r_i)h_i^2\Big(12(r_i^2 + 3r_i + 1) - r_iK^2h_i^2(6r_i^2 + r_i(4 - K^2h_i^2) + 6)\Big),$$

$$a_{0i} = 24r_i(6 - K^2h_i^2r_i^2),$$

$$c_{0i} = 2r_ih_i^2\Big(r_i(6 - r_iK^2h_i^2) + 6(r_i^2 - 1)\Big) \qquad (5)$$

Thus, together with (4) and (5), we obtained our proposed algorithm, i.e. finite difference method (3) for the approximate numerical solution of the problem (1). It is a $N \times N$ system of linear equations if forcing function $f(x, u)$ is linear otherwise system of nonlinear equations. The solution of the system of equations is the approximate numerical solution of problem (1).

IV. LOCAL TRUNCATION ERROR

To calculate the local truncation error in the algorithm (3), we apply the Taylor series expansion method at the nodal points $x = x_i$, $i = 1, 2, .., N$. Thus the truncation error T_i in method (3) may be written as:

$$T_i = a_{2i}u_{i+1} + a_{1i}u_i + a_{0i}u_{i-1} - c_{2i}f_{i+1} - c_{1i}f_i - c_{0i}f_{i-1} =$$

$$\frac{h_i^5r_i(1 - r_i)}{30}\left\{\begin{matrix} 5K^2(1 + r_i)\begin{pmatrix} 6(1 + r_i) \\ -r_i^2(6 - K^2h_i^2)\end{pmatrix}u_i^{(3)} \\ -\begin{pmatrix} 36(r_i^4 + r_i^3 + r_i^2 + r_i + 1) \\ -60(1 + r_i)(1 + r_i - r_i^3) \\ -2r_i^2K^2h_i^2(8 + 8r_i + 3r_i^2)\end{pmatrix}u_i^{(5)}\end{matrix}\right\} \qquad (6)$$

where $r_i = \frac{h_{i+1}}{h_i}$. Thus we have obtained a truncation error at each node of $O(h^5)$.

V. THE CONVERGENCE OF THE METHOD

Let us rewrite equation (1) in the following form

$$-u''(x) - K^2u(x) + f(x, u) = 0 \qquad (7)$$

Let $U_i = u(x_i)$ is exact and u_i is an approximate solution of the problem (1). Apply the algorithm (3) and at node x_i, we have

$$-a_{2i}u_{i+1} - a_{1i}u_i - a_{0i}u_{i-1} + c_{2i}f_{i+1} + c_{1i}f_i + c_{0i}f_{i-1} = 0, \quad i = 1,2,\dots,N \qquad (8)$$

The exact solution U_i of difference equation (3) will satisfy

$$-a_{2i}U_{i+1} - a_{1i}U_i - a_{0i}U_{i-1} + c_{2i}F_{i+1} + c_{1i}F_i + c_{0i}F_{i-1} + T_i = 0, \qquad i = 1,2,\ldots,N \qquad (9)$$

where T_i is the truncation error at node x_i. Also, we can redefine the forcing function in problem (1) in term of approximate and exact solutions,

$$f_i = f(x_i, u_i)$$

$$F_i = f(x_i, U_i)$$

Let we linearize the forcing functions at different nodal points and we will obtain,

$$F_i = f(x_i, U_i) = f(x_i, u_i) + (U_i - u_i)G_i,$$
$$F_{i\pm1} = f(x_{i\pm1}, U_{i\pm1}) = f(x_{i\pm1}, u_{i\pm1}) + (U_{i\pm1} - u_{i\pm1})G_{i\pm1}$$

where $G_i = \left(\frac{\partial f}{\partial u}\right)_i$ and $G_{i\pm1} = \left(\frac{\partial f}{\partial u}\right)_{i\pm1}$. Let define an error in the exact and approximate solution of the problem at different nodes i.e. $\epsilon_i = u_i - U_i$ and $\varepsilon_{i\pm1} = u_{i\pm1} - U_{i\pm1}$ Subtract (8) from (9), using the error definition we have,

$$-a_{2i}\varepsilon_{i+1} - a_{1i}\varepsilon_i - a_{0i}\varepsilon_{i-1} + c_{2i}(F_{i+1} - f_{i+1}) + c_{1i}(F_i - f_i) + c_{0i}(F_{i-1} - f_{i-1}) + T_i = 0, \quad i = 1,2,\ldots,N$$

Using (10) into above equation,

$$(a_{0i} + c_{0i}G_{i-1})\epsilon_{i-1} + (a_{1i} + c_{1i}G_i)\epsilon_i + (a_{2i} + c_{2i}G_{i+1})\epsilon_{i+1} - T_i = 0, \quad i=1,2,\ldots N \qquad (11)$$

Let us write (10) in matrix form as,

$$\boldsymbol{D\varepsilon} - \boldsymbol{T} = \boldsymbol{0} \qquad (12)$$

where

$$\boldsymbol{D} = \begin{pmatrix} a_{11}+c_{11}G_1 & a_{21}+c_{21}G_2 & & 0 \\ a_{02}+c_{02}G_1 & a_{12}+c_{12}G_2 & a_{22}+c_{22}G_3 & \\ & & \ddots & \\ 0 & & a_{0N}+c_{0N}G_{N-1} & a_{1N}+c_{1N}G_N \end{pmatrix}_{N\times N}$$

is tridiagonal matrix, $\boldsymbol{\varepsilon} = [\epsilon_1, \epsilon_2, \ldots, \epsilon_N]^T$ and $\mathbf{T} = [T_1, T_2, \ldots, T_N]^T$. Let us write $\boldsymbol{D}$ as the sum of matrices $\boldsymbol{J}$ and $\boldsymbol{E}$ i.e. $\boldsymbol{D} = \boldsymbol{J} + \boldsymbol{E}$ where

$$\boldsymbol{J} = \begin{pmatrix} a_{11} & a_{21} & & 0 \\ a_{02} & a_{12} & a_{22} & \\ & & \ddots & \\ 0 & & a_{0N} & a_{1N} \end{pmatrix}_{N\times N}$$

and

$$\boldsymbol{E} = \begin{pmatrix} c_{11}G_1 & c_{21}G_2 & & 0 \\ c_{02}G_1 & c_{12}G_2 & c_{22}G_3 & \\ & & \ddots & \\ 0 & & c_{0N}G_{N-1} & c_{1N}G_N \end{pmatrix}_{N\times N}$$

Let $m_* = \min_{1\le i\le N} K^2h_i^2$, $m^* = \max_{1\le i\le N} K^2h_i^2$ and $4.5 < m_* < m^* < 6$.

Also, let $m_* > \max_{1\le i\le N} \frac{1}{12(1+r_i)^3}$. It is easy to prove that $a_{01} > -1$, $a_{2i} > -1$ and $a_{1i} > 2$ for i = 1, 2, .., N . Also, it can be verified that $\boldsymbol{E} \ge 0$. Thus we can verify that $\boldsymbol{D} > J$. But $\boldsymbol{J}$ is invertible and moreover $\boldsymbol{J^{-1}} > 0$ [13,14]. So we have $\boldsymbol{J^{-1}} > \boldsymbol{D^{-1}}$. Let S_i denotes the sum of the elements of the i^{th} row of the matrix $\boldsymbol{J}$ where

$$S_i = \begin{cases} 24(3 + 3r_i + r_i^2)K^2h_i^2 + r_i(1+r_i)(K^2h_i^2 - 6 - 6r_i - 6r_i^2)K^4h_i^4 - 144r_i, & i = 1 \\ r_i(1+r_i)(K^4h_i^4 - 6(1 + r_i + r_i^2)K^2h_i^2 + 144), & 2 \le i \le N-1 \\ 24(3 + 3r_i + r_i^2)K^2h_i^2 + r_i(1+r_i)(K^2h_i^2 - 6 - 6r_i - 6r_i^2)K^4h_i^4 - 144r_i, & i = N \end{cases} \qquad (10)$$

Let us assume $S_* = \min_{1\le i\le N} S_i$, then we

$$\|\boldsymbol{J^{-1}}\| \le \frac{1}{S_*} \qquad (13)$$

Thus (12) and (13), we have

$$\|\boldsymbol{\varepsilon}\| \le \frac{1}{S_*}\|\boldsymbol{T}\| \qquad (14)$$

Thus, from (6) and (14) it follows that $\|\boldsymbol{\varepsilon}\| \to 0$ as $h_i \to 0$. We conclude that method (3) converges and the order of the convergence of method (3) is at least cubic.

VI. Numerical Experiments and Results

We have considered linear and nonlinear model problems to perform the numerical experiment. In each model problem, we took constant $r_i = \frac{h_{i+1}}{h_i}$. In the computation of maximum absolute error *MAEU*, we have used the following formula,

$$MAEU = \max_{1\le i\le N} |u(x_i) - u_i|$$

and in the estimation of the order of the convergence (O_N) of the method (3), we used the following formula

$$(O_N) = \log_m\left(\frac{MAEU_N}{MAEU_{(mN)}}\right)$$

where m can be estimated by considering the ratio of $N's$.

We have applied respectively Gauss-Seidel and Newton-Raphson method for the solution of systems of linear equations and non-linear equations result in the discretization of the problems using the proposed algorithm (3). The solutions are computed on N nodes and the iteration continued until either the number of iterations reached 10^2 or the maximum difference between two successive iterates is less than 10^{-8}. All the computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler

(2.95 of GCC) on Intel Core i3-2330M, 2.20 GHz PC.

Problem 1. The linear model problem given by

$$u''(x) + K^2 u(x) = K^2 x^2 + 2,\ 0 < x < 1$$

subject to boundary conditions

$u(0) = 0,$ and $u(1) = 1 + \sin(K).$

The constructed /exact analytical solution of the problem is $u(x) = \sin(Ku) + x^2$. The computed

MAEU for different values of N and RI are presented in Table 1. Also no. of iterations *Iter*. required to achieve *MAEU* presented there in Table1.

Problem 2. The linear model problem is given by

$u''(x) + K^2 u(x) = u(x) - 2K \exp(-x) \cos(Kx),$ $0 < x < 1$

subject to boundary conditions

$u(0) = 0$ and $u(1) = \exp(-1)\sin(K).$

The constructed /exact analytical solution of the problem is $u(x) = \exp(-x) \sin(Kx)$. The computed *MAEU* for different values of N and RI are presented in Table 2. Also no. of iterations *Iter*. required to achieve *MAEU* presented there in Table 2.

Problem 3. The nonlinear model problem given by

$u''(x) + K^2 u(x) = K^2 u(x)(1 - \exp(Kx)u(x) + 2 \exp(2Kx)u^2(x)),$ $0 < x < 1$

subject to boundary conditions

$$u(0) = \frac{1}{2} \quad \text{and} \quad u(1) = \frac{1.0}{1 + \exp(K)}$$

The constructed/exact analytical solution of the problem is $u(x) = \frac{1.0}{1+\exp(Kx)}$. The computed *MAEU* for different values of N and r_i are presented in Table 3. Also no. of iterations *Iter*. required to achieve *MAEU* presented there in Table 3.

Table 1: Maximum absolute error (Problem 1).

K	r_i		**N**			
			4	**8**	**16**	**32**
1.5	.85	*MAEU*	.12797039(-3)	.22780743(-4)	.72150506(-5)	.44405488(-5)
		Iter.	13	35	29	22
	.90	*MAEU*	.79272184(-4)	.10766746(-4)	.29562800(-5)	.70078812(-6)
		Iter.	12	26	27	8
	.95	*MAEU*	.43739357(-4)	.48330880(-5)	.54639708(-6)	.10074696(-5)
		Iter.	14	22	5	7
	1.00	*MAEU*	.13470650(-4)	.23841858(-6)	.59604645(-7)	.42915344(-5)
		Iter.	10	5	1	64

Table 2: Maximum absolute error (Problem 2).

K	r_i		**N**			
			4	**8**	**16**	**32**
.50	.85	*MAEU*	.25399029(-4)	.50574936(-5)	.18551946(-5)	.12591481(-5)
		Iter.	13	23	40	33
	.90	*MAEU*	.14975667(-4)	.23841858(-5)	.55134296(-6)	.17881393(-6)
		Iter.	11	23	28	13
	.95	*MAEU*	.81211329(-5)	.92387199(-6)	.59604645(-7)	.29802322(-7)
		Iter.	10	19	6	2
	1.00	*MAEU*	.20265579(-5)	.74505806(-7)	.14901161(-7)	.29802322(-7)
		Iter.	8	6	1	1

Table 3: Maximum absolute error (Problem 3).

K	r_i		**N**			
			4	**8**	**16**	**32**
2.5	.85	*MAEU*	.50046929(-4)	.86816253(-5)	.29637317(-5)	.21018782(-5)
		Iter.	6	12	20	17
	.90	*MAEU*	.36545040(-4)	.39168362(-5)	.76874841(-6)	.28658795(-6)
		Iter.	7	13	8	3
	.95	*MAEU*	.26586760(-4)	.17725698(-5)	.24027636(-6)	.94465300(-7)
		Iter.	7	12	5	14
	1.0	*MAEU*	.19564573(-4)	.11765405(-5)	.85043219(-7)	.18604103(-7)
		Iter.	6	10	6	1
3.5	.85	*MAEU*	.14678681(-3)	.36329031(-4)	.13210631(-4)	.96381737(-5)
		Iter.	5	6	12	15
	.90	*MAEU*	.10349760(-3)	.18275507(-4)	.35319379(-5)	.15788354(-5)
		Iter.	5	10	18	14
	.95	*MAEU*	.72204180(-4)	.86282880(-5)	.72915776(-6)	.28251156(-6)
		Iter.	5	9	9	13
	1.0	*MAEU*	.68829424(-4)	.41220310(-5)	.29232848(-6)	.28144444(-6)
		Iter.	6	11	6	12
4.5	.95	*MAEU*	.64944543(-4)	.25401225(-4)	.25228351(-5)	.45036066(-6)
		Iter.	7	7	13	10
	1.0	*MAEU*	.18576792(-3)	.12108090(-4)	.81300982(-6)	.91218730(-7)
		Iter.	7	8	6	5
5.5	.95	*MAEU*	.16360165(-3)	.57584149(-5)	.66526718(-5)	.11472596(-5)
		Iter.	5	10	10	17
	1.0	*MAEU*	.32609457(-3)	.26916419(-4)	.16365996(-5)	.17088547(-6)
		Iter.	4	8	14	8
6.5	.95	*MAEU*	.45986887(-3)	.75560478(-4)	.14123988(-4)	.26245261(-5)
		Iter.	7	7	8	10
	1.0	*MAEU*	.10475870(-2)	.44738903(-4)	.31817165(-5)	.32630444(-6)
		Iter.	5	6	13	7
7.5	.95	*MAEU*	.12349439(-2)	.74402131(-4)	.25594589(-4)	.51407492(-5)
		Iter.	7	8	7	10
	1.0	*MAEU*	.21523978(-2)	.96857053(-4)	.56619497(-5)	.41037129(-6)
		Iter.	4	6	11	7

Numerical results for different values of N and r_i for model problem 3 is presented in table 3. We observed from the computational results that without a change in r_i, the decrease in h_i results decreases with an *MAEU* and an increase in K results increase in *MAEU*.

As we decrease the value of r_i from .95 to less than .95 in the same value of K = 2.5 and K = 3.5, there is a substantial reduction in the order of the convergence of the proposed algorithm. Thus, we conclude that the order of the method depends on the choice of the r_i. Also, the accuracy in numerical solution u_i decreases as r_i decreases. We observed similar observations in the numerical result of problem 1 and problem 2. However, we find algorithm (3) isconvergent

VII. Conclusion

An algorithm using finite differences for the numerical solution of the Helmholtz type equation and corresponding two-point boundary value problems with non-uniform step length i.e. variable step length has developed. In the development of the algorithm, we transformed the continuous problem into a discrete problem, i.e. the differential equation problem (1) into a system of algebraic equations (3).

The proposed algorithm (3) delivers a decent approximate numerical solution of the model problems considered in numerical experiments with non-uniform/variable step size. The numerical results for the model problems showed that the proposed algorithm is computationally effective and exact. The order of convergence of the proposed algorithm depends on r_i, the ratio of step sizes. The order of the convergence very close to three as the value of r_i approaches 1. Following the thought presented in the present article, there is the possibility to develop algorithms for the numerical solution of higher-order differential equations with internal boundary condition. Works in these directions are in advancement.

REFERENCES

[1] Keller, H. B., Numerical Methods for Two-Point Boundary Value Problems. Waltham, Mass. Blaisdell Publ. Co. (1968).

[2] Ascher, U., Mattheij, R. M. M. and Russell, R. D., Numerical Solution of Boundary Value Problems for Ordinary Differential Equations. Prentice-Hall, Englewood Cliffs, NJ (1988).

[3] Baxley, J. V., Nonlinear Two-Point Boundary Value Problems in Ordinary and Partial Differential Equations (Everitt, W.N. and Sleeman, B.D. Eds.). 46-54, Springer-Verlag, New York (1981).

[4] Laird, A. L. and Giles, M. B., Preconditioned iterative solution of the 2-D Helmholtz equation. Tech. Rep.NA 02-12, Comp. Lab., Oxford Univ., 2002.

[5] Arnold, A. and Ehrhardt, M., Discrete transparent boundary conditions for wide-angle parabolic equations in underwater acoustics. J. Comput. Phys., 145, 611-638 (1998).

[6] Fix, G. J. and Marin, S. P., Variational methods for underwater acoustic problems. J. Comput. Phys., 28, 253-270 (1978)

[7] Urbach, H. P. and Merkx, R. T. M., Finite element simulation of electromagnetic plane wave diffraction at gratings for arbitrary angles of incidence. Mathematical and Numerical Aspects of Wave Propagation Phenomena (G. Cohen, L. Halpern, and P. Joly Eds.), 89-99, SIAM Publishing Company, Philadelphia (1992).

[8] Singer, I. and Turkel, E., High-order finite difference methods for the Helmholtz equation. Comput. Methods Appl. Mech. Engrg., 163, 343-358 (1998).

[9] Pandey, P. K. and Jaboob, S. S., A fourth-order finite difference method for a nonlinear Helmholtz type boundary value problems in PDEs (To have appeared).

[10] Ihlenburg, F. and Babuska, I., Finite element solution of the Helmholtz equation with high wave number part i: The h-version of the FEM. Computers & Mathematics with Applications,30, 9-37 (1995).

[11] Autrique, J. and Magoules, F., Numerical analysis of a coupled finite- in finite element method for exterior Helmholtz problems. Journal of Computational Acoustics, 14, 21-43 (2006).

[12] Erlangga, Y. A., A robust and efficient iterative method for the numerical solution of the Helmholtz equation. Doctoral thesis, Aerospace Engineering, Delft University of Technology (2005).

[13] Henrici, P., Discrete Variables Methods in Ordinary Differential Equation. John Wiley and Sons, New York (1962).

[14] Varga, R. S., Matrix Iterative Analysis, (Second Revised and Expanded Edition). Springer- Verlag, Heidelberg (2000).